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ON THE NATURE OF THE BOUNDARY LAYER AT LINES OF DISCONTINUITY OF A SURFACE LOAD ON A PLATE

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In constructing refined theories of the theory of deformation of thin elastic solids (shells, plates), its state of stress is often represented as the sum of the interior state of stress and a boundary layer [1−3]. The interior state of stress extends over the whole domain occupied by the solid, while the boundary layer state of stress is localized at the edges of the solid or at other lines of discontinuity and damps out rapidly with distance from these lines.

The nature of the boundary layer at the plate or shell edges and its interaction with the interior state of stress has been studied in sufficient detail [3, 4].

The question of the presence of a boundary layer and the comparative value of its stresses at the line of discontinuity of the surface load is considered below in an example of flexible plate bending.

A circular plate of radius b and thickness $2h$ is bent by the normal piecewise-continous load

$$p(r, \varphi) = \begin{cases} p_1(r, \varphi), & r \leqslant a \\ p_2(r, \varphi), & a \leqslant r \leqslant b \end{cases} \quad (p_1 \neq p_2)$$

The state of stress of the plate near the cylindrical surface $r = a$ will be represented in each of the domains into which this surface divides the plate as the sum of the interior state of stress and the boundary layer. We shall construct these states of stress by using fundamental and auxiliary iteration processes [3], i.e. we shall determine [4] two biharmonic functions $w^{(s)}(r, \varphi)$ and $\Phi^{(s)}(\xi, \zeta)$ and one harmonic function $\psi^{(s)}(\xi, \zeta)$ ($h\zeta = z$, $h\xi = r - a$) in each approximation for the two domains obtained in the plate.

The stress tensor and the displacement vector are continuous at each point of the section $r = a$ For $r = a$ we obtain

$$\begin{aligned}
&\sigma_{r(1)}^{(s)} + \sigma_{r[1]}^{(s)} = \sigma_{r(2)}^{(s)} + \sigma_{r[2]}^{(s)}, \qquad u_{r(1)}^{(s)} + u_{r[1]}^{(s-1)} = u_{r(2)}^{(s)} + u_{r[2]}^{(s-1)} \\
&\tau_{r\theta(1)}^{(s)} + \tau_{r\theta[1]}^{(s)} = \tau_{r\theta(2)}^{(s)} + \tau_{r\theta[2]}^{(s)}, \qquad u_{\theta(1)}^{(s)} + u_{\theta[1]}^{(s-1)} = u_{\theta(2)}^{(s)} + u_{\theta[2]}^{(s-1)} \\
&\tau_{rz(1)}^{(s-1)} + \tau_{rz[1]}^{(s)} = \tau_{rz(2)}^{(s-1)} + \tau_{rz[2]}^{(s)}, \qquad W_{(1)}^{(s)} + W_{[1]}^{(s-2)} = W_{(2)}^{(s)} + W_{[2]}^{(s-2)}
\end{aligned} \tag{1}$$

from the continuity condition by using a procedure described earlier [4] in detail.

The quantities with the subscripts (1) and (2) here refer to the interior state of stress in the domains $r \leqslant a$ and $a \leqslant r \leqslant b$, respectively, and the quantities with the subscripts [1] and [2] to the boundary layer stresses and displacements.

The arbitrariness of integrating the equations for the functions

$$W_{(i)}^{(s)}, \ \Phi_{[i]}^{(s)}, \ \Psi_{[i]}^{(s)} \qquad (i = 1, 2)$$

permits satisfying the six conditions (1) on the surface $r = a$ and the four boundary layer damping conditions in the half-strips $|\zeta| \leqslant 1, \xi \leqslant 0$ and $|\zeta| \leqslant 1, \xi \geqslant 0$ defined by the functions $\Phi_{[1]}$ and $\Phi_{[2]}$.

For the homogeneous skew symmetric problem of plane strain of a half-strip these conditions are represented as some pair of the following four equalities for each of the boundary layers [5]:

$$\int_{-1}^{1} \tau_{rz}(0, \zeta)\, d\zeta = 0, \qquad \int_{-1}^{1} \zeta\sigma_r(0, \zeta)\, d\zeta = 0 \tag{2}$$

$$\frac{\nu}{2} \int_{-1}^{1} \zeta^2 \tau_{rz}(0, \zeta)\, d\zeta + \frac{E}{1+\nu} \int_{-1}^{1} \zeta u_r^{\cdot}(0, \zeta)\, d\zeta = 0$$

$$\frac{2-\nu}{3} \int_{-1}^{1} \zeta^3 \sigma_r(0, \zeta)\, d\zeta - \frac{E}{1+\nu} \int_{-1}^{1} (\zeta^2 - 1)\, W(0, \zeta)\, d\zeta = 0$$

Depending on whether the half-strip was assumed given on the endface, these conditions have been obtained in [5] as the combination of the first and second, the first and third, or second and fourth of conditions (2).

Separating the boundary value problems of the interior state of stress and the boundary layer [4], and constructing the boundary layer, we can determine those quantities on the edge of the half-strip which have not been given there. (For example, if the damping conditions are used in the form of the combination of the first and second conditions in (2), and the boundary conditions on the edge of the half-strip are obtained as $\sigma_r(0, \zeta) = f_1(\zeta)$, $\tau_{rz}(0, \zeta) = f_2(\zeta)$, then $u_r(0, \zeta)$ and $W(0, \zeta)$ can be determined by constructing the boundary layer. The unused equalities in (2) must be satisfied identically for these quantities). This latter means that when this turns out to be necessary, all four conditions in (2) can be used to separate the boundary value problems of the interior state of stress and the boundary layer without overdetermining the problem.

We use all the conditions (2) for each of the boundary layers assumed on the surface $r = a$ ($\xi = 0$) to extract the conjugate conditions of the interior problem on the line $r = a$ from (1) for the problem under consideration.

Using conditions (1) and (2) successively and taking into account that the boundary layer stresses should satisfy [4] the homogeneous conditions at $\zeta = \pm 1$ in each approximation, we obtain the equations

$$D\Delta\Delta w_{(i)} = p_i - \frac{(8-3\nu)h^2}{10(1-\nu)} \Delta p_i \tag{3}$$

$$w = h^{-3}(w^{(0)} + hw^{(1)} + h^2 w^{(2)}), \qquad i = 1, 2$$

$$w_{(1)} = w_{(2)}, \qquad \frac{\partial w_{(1)}}{\partial r} = \frac{\partial w_{(2)}}{\partial r} \tag{4}$$

$$\frac{\partial^2 w_{(1)}}{\partial r^2} - \frac{\partial^2 w_{(2)}}{\partial r^2} = -\frac{h^2(8-3\nu)}{10D(1-\nu)}(p_1 - p_2)$$

$$\frac{\partial^3 w_{(1)}}{\partial r^3} - \frac{\partial^3 w_{(2)}}{\partial r^3} = \frac{h^2(8-3\nu)}{10D(1-\nu)}\left[\frac{p_1 - p_2}{a} - \frac{\partial}{\partial r}(p_1 - p_2)\right]$$

for the interior state of stress (with error on the order of $O(h^3)$ and the conjugate conditions at $r = a$).

For the boundary layers we have

$$\Psi_{[1]}^{(n)} = \Psi_{[2]}^{(n)} \equiv 0 \qquad (n = 0, 1, 2), \qquad \Phi_{[1]}^{(k)} = \Phi_{[2]}^{(k)} \equiv 0 \qquad (k = 0, 1) \tag{5}$$

$$\Phi_{[1]}^{(2)} = -\frac{p_1 - p_2}{4} \sum \frac{1}{\lambda_k^2 \sin^4 \lambda_k} e^{\lambda_k \xi} F_k(\zeta), \qquad -\infty < \xi \leqslant 0$$

$$\Phi_{[2]}^{(2)} = \frac{p_1 - p_2}{4} \sum \frac{1}{\lambda_k^2 \sin^4 \lambda_k} e^{-\lambda_k \xi} F_k(\zeta), \qquad 0 \leqslant \xi < \infty$$

$$F_k(\zeta) = \lambda_k \cos \lambda_k \sin \lambda_k \zeta - \lambda_k \zeta \sin \lambda_k \cos \lambda_k \zeta$$

$$\left(\sigma_{r[i]}^{(2)} = \frac{\partial^2 \Phi_{[i]}^{(2)}}{\partial \zeta^2},\ \sigma_{z[i]}^{(2)} = \frac{\partial^2 \Phi_{[i]}^{(2)}}{\partial \xi^2},\ \sigma_{\theta[i]}^{(2)} = \nu \Delta \Phi_{[i]}^{(2)},\ \tau_{rz[i]}^{(2)} = -\frac{\partial^2 \Phi_{[i]}^{(2)}}{\partial \xi\, \partial \zeta} \right)$$

Here λ_k are the roots of the equation $\sin 2\lambda - 2\lambda = 0$, which have positive real part (over which the summation in (5) is taken), and $F_k(\zeta)$ are odd Papkovich functions.

Let us analyze the results obtained. We have for the stresses of the interior state of stress

$$\sigma = h^{-q} \sum_{s=0}^{S} h^s \sigma^{(s)} \tag{6}$$

Here $q = 2$ for $\sigma_r, \sigma_\theta, \tau_{r\theta}$ and $q = 1$ for $\tau_{rz}, \tau_{\theta z}$ and $q = 0$ for σ_z. For the boundary layer stresses $q = 2$.

It follows from (5) and (6) that : (a) there is a boundary layer, a plane strain in planes perpendicular to the line $r = a$, at the line of discontinuity of the surface load ; (b) the boundary layer stresses are on the order of h° (the boundary layer stresses are on the order of h^{-2} on the plate edge, i.e. the same as in the fundamental stresses of classical theory).

Hence there results that the error of classical theory at the line of discontinuity of the surface load is of the same order of h compared to h° as far from this line.

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